The Linear Matrix-Valued Cost Functions as a Source of Leontief and Ghosh models

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A general formulation of linear input-output model is considered in the paper as a system of equations written in terms of free variables for any rectangular supply and use table given. This system spans the regular linear equations for material and financial balances with a batch of predetermined values for exogenous variables (net final demand and value added vectors).

Variations in exogenous elements of input–output model lead to the changes of price and quantity proportions in the resulting supply and use table that are formally described by two nonlinear multiplicative patterns. It is shown how these patterns can be linearized and adjusted for evaluating the input–output model at constant prices and at constant level of production.

The pattern for assessing at constant prices provides an exact identifiability of the model when the Leontief technical coefficients and the product-mix matrix are invariable. In contrast, the model based on other pattern is exactly identifiable when the Ghosh allocation coefficients and the product-mix matrix stay invariant. The regular (rectangular case) and supplementary (square case) solutions for both types of input–output models are obtained. Supplementary solutions are used to formulate generalized versions of Leontief demand-driven model and Ghosh supply-driven model.

For symmetric input-output table, the properties of diagonal production matrix allow transforming the generalized versions of Leontief and Ghosh models into the "classical" input–output models. The equivalence of Leontief price model and Ghosh supply-driven model as well as the equivalence of Leontief demand-driven model and Ghosh quantity model is proven. It is to be noted that relevant formulas do demonstrate a remarkable set of duality properties.

Keywords: rectangular supply and use table, exogenous changes in final demand and value added, matrix-valued cost functions, demand-driven and supply-driven models, price and quantity models

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1. A general formulation of linear input–output model

The general linear input–output model of an economy with N of products (commodities) and M industries (sectors) for the certain time period leans on a pair of rectangular matrices, namely supply (production) matrix $\mathbf{X}$ and use for intermediates (intermediate consumption) matrix $\mathbf{Z}$ of the same dimension $N\times M$ both. In mathematical notation, the model includes the vector equation for material balance of products' intermediate and final uses, i.e.,

$$\mathbf{X}\mathbf{e}_M = \mathbf{Z}\mathbf{e}_M + \mathbf{y}\,, \tag{1}$$

and the following vector equation for financial balance of industries' intermediate and primary (combined into value added) inputs:

$$\mathbf{e}'_N\mathbf{X} = \mathbf{e}'_N\mathbf{Z} + \mathbf{v}' \tag{2}$$

where $\mathbf{e}_N$ and $\mathbf{e}_M$ are $N\times 1$ and $M\times 1$ summation column vectors with unit elements, $\mathbf{y}$ is a column vector of net final demand with dimensions $N\times 1$, and $\mathbf{v}$ is a column vector of value added with dimensions $M\times1$. Here putting a prime after vector's (matrix's) symbol denotes a transpose of this vector (matrix).

"One of the major uses of the information in an input–output model is to assess the effect on an economy of changes in elements that are exogenous to the model of that economy" (Miller and Blair, 2009, p. 243). To measure the changes mentioned above, in most practical cases there usually is the supply and use table for economy under consideration for the same time period (say, period 0) compiled from available statistical data. This table includes the production matrix X_0 and intermediate consumption matrix $\mathbf{Z}_0$ with dimensions $N\times M$, ($N\times1$)-dimensional column vector of net final demand $\mathbf{y}_0$, and ($M\times1$)-dimensional column vector of value added $\mathbf{v}_0$ (see Eurostat, 2008). Note that the equations (1) and (2) are exactly met for the initial table components.

2. The price and quantity transformations of variables in the input–output model

With accordance to the quotation above, the main aim of constructing input–output models is to assess an impact of the exogenous changes (either absolute or relative) in net final demand and, by virtue of symmetry in the balance equations under consideration, the exogenous changes in gross value added on simultaneous behavior of the economy. Balance models do not usually reflect the true causes of the certain changes in final demand or value added, so the response of the economy to any exogenous disturbance is evaluated in the mode of getting answers to questions like "what would happen if ...? ".

In principle, variations in exogenous elements of the input–output model (1), (2) lead to the changes of price and quantity proportions in the resulting supply and use table. The most general way to describe an impact of these changes on matrices X and Z is as follows:

$$\mathbf{X} = \mathbf{P_X} \circ \mathbf{Q_X} \circ \mathbf{X}_0, \qquad \mathbf{Z} = \mathbf{P_Z} \circ \mathbf{Q_Z} \circ \mathbf{Z}_0$$

where $\mathbf{P_X}$ and $\mathbf{P_Z}$ are $N\times M$-dimensional matrices of the relative price indices for products, $\mathbf{Q_X}$ and $\mathbf{Q_Z}$ are $N\times M$ matrices of the relative quantity (physical volume) indices for industries of the economy, and the character " $\circ$ " denotes the Hadamard's (element-wise) product of two matrices with the same dimensions.

One can assume that in market economy $\mathbf{P_X} = \mathbf{P_Z} = \mathbf{P}$, and $\mathbf{Q_X} = \mathbf{Q_Z} = \mathbf{Q}$ on the current level of production. Besides, it is quite natural to propose also that the price on certain product does not vary along the row of producing-and-consuming industries, i.e., $p_{nm} = p_n$ for all $m = 1\div M$ at $n = 1\div N$ where the character " $\div$ " between the lower and upper bounds of index's changing range means that the index sequentially runs all integer values in the specified range, and, moreover, that the production quantity index for the certain industry's output and intermediate consumption is keeping invariable through all products produced and consumed, namely, $q_{nm} = q_m$ for all $n = 1 \div N$ at $m = 1 \div M$.

Thus, matrices **P** and **Q** can be represented respectively as $\mathbf{P} = \mathbf{p} \otimes \mathbf{e}'_M$ and $\mathbf{Q} = \mathbf{e}_N \otimes \mathbf{q}'$ where **p** is a column vector of the relative price indices on products with dimensions $N \times 1$, **q** is a column vector of the relative quantity indices for industries with dimensions $M \times 1$, and the character " $\otimes$ " denotes the Kronecker product for two matrices.

Transforming the above statements into regular matrix notation gives two nonlinear multiplicative patterns

$$\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0\hat{\mathbf{q}} \tag{3}$$

where putting a "hat" over vector's symbol (or angled bracketing around it) denotes a diagonal matrix with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697). The patterns (3) provide the combined price and quantity description of an economy response to exogenous changes in the input–output model's variables, inter alia, in net final demand and in gross value added. Note that vectors **p** and **q** in (3) cannot be estimated unambiguously because the patterns (3) are hyperbolically homogeneous, since $\mathbf{X} = \mathbf{p}\mathbf{q}' \circ \mathbf{X}_0$ and $\mathbf{p}\mathbf{q}' = c\mathbf{p} \cdot \mathbf{q}'/c$ for any nonzero scalar c.

Nevertheless, evaluations of input–output model (1), (2) in terms of the production quantity changing at constant prices on the products and/or in terms of price changing at constant level of production in the industries are of great theoretical and practical interest.

3. Evaluating the input–output model at constant prices

In a case of constant prices on products we have $\hat{\mathbf{p}} = \mathbf{E}_N$ where $\mathbf{E}_N$ is identity matrix of order N, so the nonlinear multiplicative patterns (3) can be rewritten in linear form, namely

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}, \qquad \mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}. \tag{4}$$

According to the first equation (4), the row vector of industry outputs is equal to $\mathbf{e}'_N\mathbf{X} = \mathbf{e}'_N\mathbf{X}_0\hat{\mathbf{q}} = \mathbf{q}'\langle\mathbf{e}'_N\mathbf{X}_0\rangle$ from which

$$\hat{\mathbf{q}} = \langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\langle\mathbf{e}'_N\mathbf{X}\rangle$$

where the obvious commutativity property of diagonal matrices is used. Substituting the latter expression in multiplicative patterns (4) gives two matrix-valued linear functions

$$\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}} = \mathbf{X}_0\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\langle\mathbf{e}'_N\mathbf{X}\rangle = \mathbf{G}\langle\mathbf{e}'_N\mathbf{X}\rangle, \tag{5}$$

$$\mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}} = \mathbf{Z}_0\langle\mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\langle\mathbf{e}'_N\mathbf{X}\rangle = \mathbf{A}\langle\mathbf{e}'_N\mathbf{X}\rangle \tag{6}$$

with vector of industry outputs $\mathbf{e}'_N\mathbf{X}$ as their mutual argument. Note that matrix

$\mathbf{G} = \mathbf{X}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}$ is known in special literature as product-mix matrix (see Eurostat, 2008) with shares of each product in output of an industry in a column. The matrix $\mathbf{G}$ in (5) provides a linkage between production matrix $\mathbf{X}$ and its column marginal totals.

Function (6) establishes a linear dependence of intermediate consumption matrix $\mathbf{Z}$ from the industry outputs $\mathbf{e}'_N \mathbf{X}$, and so it can be classified as *the matrix-valued linear cost function*. Matrix $\mathbf{A} = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1}$ is widely known under the name of (Leontief) technical coefficients matrix (see, e.g., Miller and Blair, 2009).

Substituting multiplicative patterns (4) in the input–output model (1), (2), we obtain

$$(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{q} = \mathbf{y}, \tag{7}$$

$$\langle \mathbf{e}'_N (\mathbf{X}_0 - \mathbf{Z}_0) \rangle \mathbf{q} = \hat{\mathbf{v}}_0 \mathbf{q} = \mathbf{v} \tag{8}$$

respectively. Further, as it follows from (5) and (6), matrices $\mathbf{A}$ and $\mathbf{G}$ stay invariant in the process of evaluating the input–output model at constant prices. That is the reason why below we will call the linear equations (7) and (8) by the model AG.

4. Evaluating the input–output model at constant level of production

Assessing the model (1), (2) at constant level of production in the industries (at $\hat{\mathbf{q}} = \mathbf{E}_M$ where $\mathbf{E}_M$ is identity matrix of order M) leads to linear patterns

$$\mathbf{X} = \hat{\mathbf{p}} \mathbf{X}_0, \qquad \mathbf{Z} = \hat{\mathbf{p}} \mathbf{Z}_0. \tag{9}$$

In accordance with the first equation (9), the column vector of product outputs is equal to $\mathbf{X}\mathbf{e}_M = \hat{\mathbf{p}} \mathbf{X}_0 \mathbf{e}_M = \langle \mathbf{X}_0 \mathbf{e}_M \rangle \mathbf{p}$ from which

$$\hat{\mathbf{p}} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \langle \mathbf{X}\mathbf{e}_M \rangle.$$

Substituting the latter expression in multiplicative patterns (9) gives two other matrix-valued linear functions

$$\mathbf{X} = \hat{\mathbf{p}} \mathbf{X}_0 = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0 = \langle \mathbf{X}\mathbf{e}_M \rangle \mathbf{H}, \tag{10}$$

$$\mathbf{Z} = \hat{\mathbf{p}} \mathbf{Z}_0 = \langle \mathbf{X}\mathbf{e}_M \rangle \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0 = \langle \mathbf{X}\mathbf{e}_M \rangle \mathbf{B} \tag{11}$$

with vector of product outputs $\mathbf{X}\mathbf{e}_M$ as their mutual argument. Note that matrix $\mathbf{H} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{X}_0$ is known in literature as market shares matrix (see Eurostat, 2008) with contributions of each industry to the output of a product in a row. The matrix $\mathbf{H}$ in (10) provides a linkage between production matrix $\mathbf{X}$ and its row marginal totals.

Function (11) establishes a linear dependence of intermediate consumption matrix $\mathbf{Z}$ from the product outputs $\mathbf{X}\mathbf{e}_M$, and so it can be also classified as *the matrix-valued linear cost* *function*. Matrix $\mathbf{B} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \mathbf{Z}_0$ is known under the name of (Ghosh) allocation coefficients matrix (see, e.g., Miller and Blair, 2009).

Finally, substituting multiplicative patterns (9) in the input–output model (1), (2), we have

$$\langle (\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_M \rangle \mathbf{p} = \hat{\mathbf{y}}_0 \mathbf{p} = \mathbf{y}, \tag{12}$$

$$(\mathbf{X}'_0 - \mathbf{Z}'_0)\mathbf{p} = \mathbf{v} \tag{13}$$

respectively. As it follows from (10) and (11), matrices **B** and **H** stay invariant in the process of evaluating the input–output model at constant level of production. That is the reason why below we will call the linear equations (12) and (13) by the model BH.

5. Regular and supplementary solutions for the model AG

Consider some operational opportunities in obtaining solutions for the model AG (7), (8) in the cases of evaluating a response of the economy to exogenous changes in the net final demand vector $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ with dimensions $N\times 1$ or in the value added vector $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ with dimensions $M\times 1$. Here it is assumed that "disturbed" vectors $\mathbf{y}_*$ and $\mathbf{v}_*$ do not have any zero components.

The material balance model (7) contains N linear equations with M scalar variables **q**, whereas the financial balance model (8) includes M linear equations with the same M unknowns. Hence, in a very general case $N \neq M$ one can assess a response of the economy only to exogenous change in the value added vector $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ by resolving the equation (8) written as $\langle \mathbf{e}'_N (\mathbf{X}_0 - \mathbf{Z}_0) \rangle \mathbf{q} = \hat{\mathbf{v}}_0 \mathbf{q} = \mathbf{v}_*$ with respect to the column vector of the relative quantity indices for industries, namely

$$\mathbf{q} = \hat{\mathbf{v}}_0^{-1} \mathbf{v}_*. \tag{14}$$

It should be noted that the solution (14) is valid at any numbers of products and industries in the economy. Nevertheless, this regular solution is trivial because a response of model AG to the disturbance $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$ comes to the alternate multiplying the columns of production and intermediate consumption matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ on the growth indices of value added through all industries at constant prices on the products.

However, at $N = M = K$ a choice of alternative exogenous condition is also feasible in finding a supplementary solution for the model AG. Under the exogenous final demand condition $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$, the equation (8) written as $(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{q} = \mathbf{y}_*$ can be resolved with respect to the column vector of the relative quantity indices for industries, namely

$$\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1} \mathbf{y}_*, \tag{15}$$

of course, if an inverse of the square (at $N = M = K$) matrix $\mathbf{X}_0 - \mathbf{Z}_0$ exists as it is expected to be. (Note that initial production matrix $\mathbf{X}_0$ usually has the dominant main diagonal.) The supplementary solution (15) is valid only if the values of N and M coincide, but it is not trivial in contrast to regular solution (14).

6. Regular and supplementary solutions for the model BH

In its turn, consider operational opportunities in getting solutions for the model BH (12), (13) in the cases of evaluating a response of the economy to exogenous changes in the final demand vector $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ or in the value added vector $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$.

The material balance model (12) contains N linear equations with N scalar variables $\mathbf{p}$, whereas the financial balance model (13) includes M linear equations with the N unknowns. Hence, in a general case $N \neq M$ one can evaluate a response of the economy only to exogenous change in the final demand vector $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ by resolving the equation (12) written as $\langle (\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{e}_M \rangle \mathbf{p} = \hat{\mathbf{y}}_0 \mathbf{p} = \mathbf{y}_*$ with respect to the column vector of the relative price indices on products, namely

$$\mathbf{p} = \hat{\mathbf{y}}_0^{-1} \mathbf{y}_* . \tag{16}$$

The regular solution (16) is valid at any numbers of products and industries in the economy. Nevertheless, this solution is trivial because a response of model BH to the disturbance $\mathbf{y} = \mathbf{y}_* \neq \mathbf{y}_0$ comes to the alternate multiplying the rows of production and intermediate consumption matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ on the value indices of final demand through all products at constant level of production in the industries.

However, at $N = M = K$ a choice of alternative exogenous condition is also feasible in finding a supplementary solution for the model BH. Under the exogenous value added condition $\mathbf{v} = \mathbf{v}_* \neq \mathbf{v}_0$, the equation (13) written as $(\mathbf{X}'_0 - \mathbf{Z}'_0)\mathbf{p} = \mathbf{v}_*$ can be resolved with respect to the column vector of the relative price indices on products, namely

$$\mathbf{p} = (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1} \mathbf{v}_* , \tag{17}$$

of course, if an inverse of the square (at $N = M = K$) matrix $\mathbf{X}'_0 - \mathbf{Z}'_0$ exists. (Recall that initial production matrix $\mathbf{X}_0$ usually has the dominant main diagonal.) The supplementary solution (17) is valid only if the values of N and M coincide, but in contrast to the regular solution (16), it is not trivial.

It is interesting here to pay attention to the fact that models AG and BH do demonstrate a remarkable set of duality properties in pairwise comparison of the regular solutions (14) and (16) as well as the supplementary solutions (15) and (17) at $N = M = K$.

7. Generalizing Leontief demand-driven model and Ghosh supply-driven model

The model AG and its supplementary solution (15) together with the resulting disturbances in production and intermediate consumption matrices (4) describe an impact of exogenous changes in final demand in terms of the production quantity changing at constant prices on the products. The model BH and its supplementary solution (17) together with the resulting disturbances in production and intermediate consumption matrices (9) characterize an impact of exogenous changes in value added in terms of price changing at constant level of production in the industries.

Model AG at $N = M = K$ can be considered as a generalized version of well-known Leontief demand-driven model (see Miller and Blair, 2009, Section 2.2.2). It serves to assess an impact of exogenous (absolute or relative) changes in final demand on the economy at constant prices. Indeed, as it follows from (4), the main fundamentals of model AG are $\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}$ and $\mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}$ where

$$\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = \left[\langle \mathbf{X}_0\mathbf{e}_K \rangle (\mathbf{H} - \mathbf{B})\right]^{-1}\mathbf{y}_* = (\mathbf{H} - \mathbf{B})^{-1}\langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}\mathbf{y}_* \tag{18}$$

according to (15). Total requirements matrix, which links the vector of product outputs with the final demand vector, can be derived as follows:

$$\mathbf{X}\mathbf{e}_K = \mathbf{X}_0\mathbf{q} = \mathbf{X}_0(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = \left[(\mathbf{X}_0 - \mathbf{Z}_0)\mathbf{X}_0{}^{-1}\right]^{-1}\mathbf{y}_* = \left(\mathbf{E}_K - \mathbf{Z}_0\mathbf{X}_0{}^{-1}\right)^{-1}\mathbf{y}_*. \tag{19}$$

Model BH at $N = M = K$ can be classified as a generalized version of Ghosh supply-driven model (see Miller and Blair, 2009, Section 12.1). It helps to evaluate an impact of exogenous (absolute or relative) changes in value added on the economy at fixed production scales (at constant level of production). As it follows from (9), the main fundamentals of model BH are $\mathbf{X} = \hat{\mathbf{p}}\mathbf{X}_0$ and $\mathbf{Z} = \hat{\mathbf{p}}\mathbf{Z}_0$ where

$$\mathbf{p} = (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1}\mathbf{v}_* = \left[\langle \mathbf{e}'_K\mathbf{X}_0 \rangle (\mathbf{G}' - \mathbf{A}')\right]^{-1}\mathbf{v}_* = (\mathbf{G}' - \mathbf{A}')^{-1}\langle \mathbf{e}'_K\mathbf{X}_0 \rangle^{-1}\mathbf{v}_* \tag{20}$$

in accordance with (17). A Ghosh analogue of total requirements matrix, which links the vector of industry outputs with the value added vector, can be derived as follows:

$$\mathbf{X}'\mathbf{e}_K = \mathbf{X}'_0\mathbf{p} = \mathbf{X}'_0(\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1}\mathbf{v}_* = \left[(\mathbf{X}'_0 - \mathbf{Z}'_0)(\mathbf{X}'_0)^{-1}\right]^{-1}\mathbf{v}_* = \left[\mathbf{E}_K - \mathbf{Z}'_0(\mathbf{X}'_0)^{-1}\right]^{-1}\mathbf{v}_*. \tag{21}$$

Here it is worth to mention the duality properties of models AG and BH again, because a response of model AG to the disturbance of the final demand coefficients $\langle \mathbf{X}_0\mathbf{e}_M \rangle^{-1}\mathbf{y}_*$ is described in the equation (18) by matrices **H** and **B**, whereas a response of model BH to the disturbance of the value added coefficients $\langle \mathbf{e}'_N\mathbf{X}_0 \rangle^{-1}\mathbf{v}_*$ is represented in the equation (20) by matrices **G** and **A**.

8. The Leontief and Ghosh models for symmetric input–output table

Vectors **q** and **p** are defined in Section 7 under an assumption that all the matrices in (18) – (21) are square (at $N = M = K$). In addition, let the initial production matrix $\mathbf{X}_0$ be a diagonal one as in a symmetric input–output table. Then the generalized versions of Leontief and Ghosh models considered above can be easily led to a "classical" view.

For diagonal matrix $\mathbf{X}_0$ of order K,

$$\mathbf{X}_0 = \mathbf{X}'_0 = \langle \mathbf{e}'_K \mathbf{X}_0 \rangle = \langle \mathbf{X}_0 \mathbf{e}_K \rangle . \quad (22)$$

The most famous Leontief formula can be obtained using (18), (22) and some algebraic properties of diagonal matrices along the sequential transformations of the product outputs vector $\mathbf{Xe}_K$ as follows:

$$\mathbf{Xe}_K = \mathbf{X}_0\mathbf{q} = \mathbf{X}_0(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = \left[(\mathbf{X}_0 - \mathbf{Z}_0)\langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1}\right]^{-1}\mathbf{y}_* = (\mathbf{E}_K - \mathbf{A})^{-1}\mathbf{y}_* \quad (23)$$

where **A** is the (Leontief) technical coefficients matrix.

Its analogue for Ghosh supply-driven model can be easily derived in the similar manner, using (20) and then (22) along the sequential transformations of the industry outputs vector $\mathbf{X}'\mathbf{e}_K$ as follows:

$$\mathbf{X}'\mathbf{e}_K = \mathbf{X}'_0\mathbf{p} = \mathbf{X}'_0(\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1}\mathbf{v}_* = \left[(\mathbf{X}'_0 - \mathbf{Z}'_0)\langle \mathbf{X}'_0 \mathbf{e}_K \rangle^{-1}\right]^{-1}\mathbf{v}_* = (\mathbf{E}_K - \mathbf{B}')^{-1}\mathbf{v}_* \quad (24)$$

where **B** is the (Ghosh) allocation coefficients matrix.

It is to be emphasized that direct putting (22) into the main statement for generalized version of Ghosh supply-driven model (20) gives well-known formula

$$\mathbf{p} = (\mathbf{X}'_0 - \mathbf{Z}'_0)^{-1}\mathbf{v}_* = (\langle \mathbf{e}'_K \mathbf{X}_0 \rangle - \mathbf{Z}'_0)^{-1}\mathbf{v}_* = (\mathbf{E}_K - \mathbf{A}')^{-1}\langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1}\mathbf{v}_* \quad (25)$$

for so-called Leontief price model (see Miller and Blair, 2009, p. 44). Thus, in the case of a symmetric input-output table (when $\mathbf{X}_0$ is diagonal matrix) the Ghosh supply-driven model coincides with the Leontief price model (see Dietzenbacher, 1997).

It can be shown in similar manner that the Leontief demand-driven model serves as the Ghosh quantity model. Indeed, direct substituting (22) into the main statement for generalized version of Leontief demand-driven model (18) gives

$$\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y}_* = (\langle \mathbf{X}_0 \mathbf{e}_K \rangle - \mathbf{Z}_0)^{-1}\mathbf{y}_* = (\mathbf{E}_K - \mathbf{B})^{-1}\langle \mathbf{X}_0 \mathbf{e}_K \rangle^{-1}\mathbf{y}_* . \quad (26)$$

It is appropriate to mention that all formulas obtained above in this and previous section demonstrate a remarkable set of duality properties, especially in pairwise comparison of (19) and (23), (21) and (24), (23) and (25), (24) and (26).

9. Concluding remarks

A general formulation of linear input-output model is considered in the paper as a system of equations written in terms of free variables for any rectangular supply and use table given. This system spans the regular linear equations for material and financial balances with a batch of predetermined values for exogenous variables (net final demand and value added vectors).

Variations in exogenous elements of input–output model lead to the changes of price and quantity proportions in the resulting supply and use table that are formally described by the nonlinear multiplicative patterns (3). These patterns can be adjusted for evaluating the input–output model at constant prices in linear form (4) and at constant level of production in linear form (9).

The proposed approach for assessing at constant prices provides an exact identifiability of the model within rectangular and square formats in the circumstances when the Leontief technical coefficients and the product-mix matrix are invariable. Under prescription for level of production to be constant, the input–output model is exactly identifiable within both formats when the Ghosh allocation coefficients and the product-mix matrix stay invariant. The regular and supplementary solutions for models AG and BH are obtained in (14) – (17). Square models AG and BH that are based on supplementary solutions (15) and (17) can be classified as generalized versions of Leontief demand-driven model and Ghosh supply-driven model respectively.

In a case of symmetric input-output table, the properties of diagonal production matrix allow transforming the generalized versions of Leontief and Ghosh models into the “classical” input–output models. In this context, the equivalence of Leontief price model and Ghosh supply-driven model as well as the equivalence of Leontief demand-driven model and Ghosh quantity model is proven. It is interesting to note that relevant formulas do demonstrate a remarkable set of duality properties.

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